1. Find a closed form formula for the sequence (a_n) defined by $a_0 = 2, a_1 = 12$, and $a_{n+2} = 7a_{n+1} - 10a_n$ for $n \geq 0$.

   First we rewrite the equation using the advancement operation:

   $$0 = (A^2 - 7A + 10)a$$
   $$0 = (A - 2)(A - 5)a$$

   The general solution will have the form $a_n = x2^n + y5^n$. Plugging in the initial conditions we see $x + y = 2$ and $2x + 5y = 12$, thus $x = -\frac{2}{3}$, $y = \frac{8}{3}$.

   The closed form of a_n is $a_n = -\frac{2}{3}2^n + \frac{8}{3}5^n$.

2. Find a closed form formula for the sequence (a_n) defined by $a_0 = 1, a_1 = 3, a_2 = 27$, and $a_{n+3} = 9a_{n+2} - 27a_{n+1} + 27a_n$ for $n \geq 0$.

   First we rewrite the equation using the advancement operation

   $$0 = (A^3 - 9A^2 + 27A - 27)a$$
   $$0 = (A - 3)^3 a$$

   This equation has a solution with the form $a_n = c3^n$ and the general solution will have the form $a_n = c_1 3^n + c_2 n 3^n + c_3 n^2 3^n$.

   Plugging in the initial conditions, we get the following system of equations,

   $$c_1 = 1$$
   $$3 = 3c_1 + 3c_2 + 3c_3$$
   $$27 = 9c_1 + 18c_2 + 36c_3$$

   Solving for the unknowns, we find: $c_1 = 1, c_2 = -1, c_3 = 1$.

   Thus, the closed form is: $a_n = 3^n - n3^n + n^2 3^n$.

3. Give the general solution to the recurrence $a_{n+3} = 4a_{n+2} - 5a_{n+1} + 2a_n$. (Note: $x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2)$)

First, we rewrite the recurrence using the advancement operator:

$$0 = (A^3 - 4A^2 + 5A - 2)a$$
$$0 = (A-1)^2(A-2)a$$

This equation has general solutions of the form

$$a_n = c_1 + nc_2 + c_3 2^n$$

4. Let p be a polynomial of degree k. Suppose λ is not a root of p. Find a solution to the recurrence $(p(A)a)(n) = \lambda^n$.

We would like to find a particular solution to the nonhomogeneous recurrence,

$$f_0(n) = d\lambda^n$$
$$p(A)d\lambda^n = \lambda^n$$
$$(\sum A^k c^k)d\lambda^n = \lambda^n$$
$$d\sum c^k A^k \lambda^n = \lambda^n$$
$$d\sum c^k \lambda^{n+k} = \lambda^n$$
$$d\sum c^k \lambda^k = 1$$
$$dp(\lambda) = 1$$
$$d = \frac{1}{p(\lambda)}$$

Thus, a particular solution will be,

$$f_0(n) = \frac{\lambda^n}{p(\lambda)}$$

5. Let α and β be nonzero and distinct. Find the general solution to the recurrence

$$a_{n+2} - (\alpha + \beta)a_{n+1} + \alpha\beta a_n = \alpha^n.$$

First, we rewrite the recurrence using the advancement operator:

$$(A^2 - (\alpha + \beta)A + \alpha\beta)a = \alpha^n$$
$$(A - \alpha)(A - \beta) = \alpha^n$$

The general solution to the nonhomogeneous recurrence will be of the form $f = f_0 + f_1$ where f_0 is a particular solution and f_1 is a solution to the corresponding homogeneous recurrence. The homogeneous recurrence will be,

$$(A - \alpha)(A - \beta) = 0$$

So it will have a general solution with the form,

$$f_1(n) = c_1(\alpha^n) + c_2(\beta^n)$$

Since α^n is a solution the homogeneous equation, we choose a particular solution of the form,

$$f_0(n) = dn\alpha^n$$
$$\implies (A - \beta)(A - \alpha)dn\alpha^n = (A - \beta)(d(n+1)\alpha^{n+1} - dn\alpha^{n+1})$$
$$= (A - \beta)d\alpha^{n+1}$$
$$= d\alpha^{n+2} - d\alpha^{n+1}\beta$$
$$= \alpha^n(d\alpha^2 - d\beta\alpha)$$

We want this to equal α^n so we let $d = \frac{1}{\alpha(\alpha-\beta)}$. Therefore the particular solution will be,

$$f_0(n) = \frac{n\alpha^n}{\alpha(\alpha - \beta)}$$

And the general solution will be,

$$f(n) = \frac{n\alpha^n}{\alpha(\alpha - \beta)} + c_1(\alpha^n) + c_2(\beta^n)$$

6. Find the general solution to the recurrence $a_{n+2} - 5a_{n+1} + 6a_n = n + 1$.

First, we rewrite the recurrence using the advancement operator:

$$(A^2 - 5A + 6)a = n + 1$$

The corresponding homogeneous equation is,

$$(A^2 - 5A + 6)a = 0$$
$$(A - 3)(A - 2) = 0.$$

Which will have solutions of the form,

$$f_1(n) = c_1 3^n + c_2 2^n$$

The particular solutions will have the form,

$$f_0(n) = c_3 + c_4 n$$
$$(A - 3)(A - 2)(c_3 + c_4 n) = n + 1$$
$$(A - 3)(c_3 + c_4(n + 1) - 2c_3 - 2c_4 n) = n + 1$$
$$n(2c_4) + (2c_3 - 3c_4) = n + 1$$
$$\implies c_4 = \frac{1}{2}$$
$$\implies c_3 = \frac{5}{4}$$
$$\implies f_0(n) = \frac{5}{4} + \frac{1}{2}n$$

So the general solution will have the form,

$$f(n) = c_1 3^n + c_2 2^n + \frac{1}{2}n + \frac{5}{4}$$

7. Let p be a polynomial with $p(1) \neq 0$. Let g be a polynomial of degree $\leq d$. Show that there is a unique polynomial f of degree $\leq d$ such that the sequence $(f(n))$ satisfies the recursion $p(A)f = g$.

Using the hint, we let P_d denote the space of polynomial sequences of degree $\leq d$, i.e., P_d consists of sequences $(f(n))$ where f is a polynomial of degree $\leq d$.

We may assume that P_d is a vector space and since p is a polynomial, we can also assume that $p(A)$ is linear. Then $\forall f_d \in P_d$,

$$f_d(x) = c_0 + c_1 x + c_2 x^2 + ... + c_d x^d$$

We can see that the basis of the space of all polynomial sequences of this form will be $\{1, x, x^2, ..., x^d\}$. Since the basis is finite, it must be true that P_d is finite dimensional.

We can also observe that the transformation $p(A)$ goes from P_d to P_d, that is, the dimension of a function will never exceed d. $\forall f_d \in P_d$ we've seen $f_d(n)$ is a linear combination of terms of n with degrees less than or equal to d. Applying the advancement operator on any individual term of degree $l \leq d$ will yield, $A(n^l) = (n+1)^l = n^l + ... + 1$. Thus A sends all terms from $P_d \to P_d$. Using linearity, we can see that applying A to all linear combinations of a polynomial will also be sent to P_d. We can see that any powers of A will be sent to P_d as well since the leading term will never be able to exceed degree d.

Next, we aim to show that $p(A)$ gives a bijective linear transformation $P_d \to P_d$. Since P_d is finite dimensional and $p(A) : P_d \to P_d$, we know from linear algebra that $p(A)$ is bijective $\iff$ $p(A)$ is injective $\iff$ $p(A)$ is surjective. Thus we can prove bijectivity by proving injectivity.

To show injectivity, we aim to show that $ker(p(A)) = 0$ (i.e., the kernel is trivial).

$$p(A)(\sum_{j=0}^{l} a_j n^j) = (\sum_{i=1}^{k} c_i A^i)(\sum_{j=0}^{l} a_j n^j)$$

We know $a_l \neq 0$. We want to find the coefecient of n^l so we can calculate

$$(\sum_{i=1}^{k} c_i A^i) a_l n^l = a_l (\sum_{i=1}^{k} c_i A^i) n^l$$
$$An^l = A(n^l + [\text{terms of degree} < l])$$
$$= (n+1)^l + [\text{terms of degree} < l]$$
$$= n^l + [\text{terms of degree} < l]$$

Again, we see that applying the advancement operator several times will yield something of the form $n^l + [\text{terms of degree} < l]$. Moreover, the total coefecient of the n^l term will be sum of the coefficients of A, i.e. $\sum_{i=1}^{k} c_i$. As long as this sum is not zero, the polynomial will not be zero and the kernel is trivial. We can use the fact that $p(1) \neq 0$ to see this the sum is not zero and the kernel of $p(A)$ is trivial.

Using the previous findings we can conclude that the transformation $p(A)$ is a bijection from $P_d \to P_d$. Thus for all g there exists a unique polynomial f such that the sequence $(f(n))$ satisfies the recursion $p(A)f = g$.

8. Find a solution to the recursion $((A-1)(A-2)a)(n) = n + 2 \cdot 3^n$.

First, we solve the corresponding homogeneous recursion, $(A-1)(A-2)a = 0$ which will have solutions of the form

$$f_1(n) = c_1 + c_2 2^n$$

We may split the particular solution into to parts and solve them separately, then by linearity we can add them back together to get the final general solution.

$$f_0^{(1)}(n) = c_3 n^2 + c_4 n + c_5$$
$$(A^2 - 3A + 2)(c_3 n^2 + c_4 n + c_5) = n$$
$$2c_3 n^2 + 2c_4 n - 3c_3(n+1)^2 + c_3(n+2)^2 - 3c_4(n+1) + c_4(n+2) = n$$
$$-2c_3 n + c_3 - c_4 = n$$
$$\implies c_3 = -\frac{1}{2}$$
$$\implies c_4 = \frac{1}{2}$$
$$\implies c_5 = 0$$

Thus, the first part of the particular solution will have the form,

$$f_0^{(1)}(n) = -\frac{1}{2}n^2 + \frac{1}{2}n$$

For the second part of the particular solution we try,

$$f_0^{(2)}(n) = d3^n$$
$$(A^2 - 3A + 2)d3^n = 2 \cdot 3^n$$
$$(3^{n+1} - 3 \cdot 3^{n+1} + 2 \cdot 3^n)d = 2 \cdot 3^n$$
$$d(9 - 9 + 2) = 2 \cdot 1$$
$$d = 1$$
$$\implies f_0^{(2)}(n) = 3^n$$

Thus, the general solution will be,

$$f(n) = c_1 + c_2 2^n + -\frac{1}{2}n^2 + \frac{1}{2}n + 3^n$$

9. There is a famous puzzle called the Towers of Hanoi that consists of three pegs and n circular discs, all of different sizes. The discs start on the leftmost peg, with the largest disc on the bottom, the second largest on top of it, and so on, up to the smallest disc on top. The goal is to move the discs so that they are stacked in this same order on the rightmost peg. However, you are allowed to move only one disc at a time, and you are never able to place a larger disc on top of a smaller disc. Let t_n denote the fewest moves (a move being taking a disc from one peg and placing it onto another) in which you can accomplish the goal. Determine an explicit formula for t_n.

To move the largest disc to the rightmost peg, we must first move all other discs onto the middle peg first. This can be done in t_{n-1} moves. Next we add one to move the largest disc to the right peg and again repeat moving all the discs from the centre peg to the right peg in t_{n-1} moves. This results in the following recursion,

$$t_n = 2t_{n-1} + 1, t_1 = 1; n > 0$$

To find a closed form formula, we rewrite the equation using the advancement operation:

$$(A - 2) = 1$$

We see that the general solution will have the form

$$f(n) = c_1 2^n + c_2.$$

Solving for c_2,

$$2c_2 + 1 = 1$$
$$\implies c_2 = -1.$$

Solving for c_1,

$$f(1) = c_1 (2)^1 - 1 = 1$$
$$\implies c_1 = 1$$

Thus, an explicit formula is given by:

$$t_n = 2^n - 1$$